Review of Eigenvectors and Eigenvalues

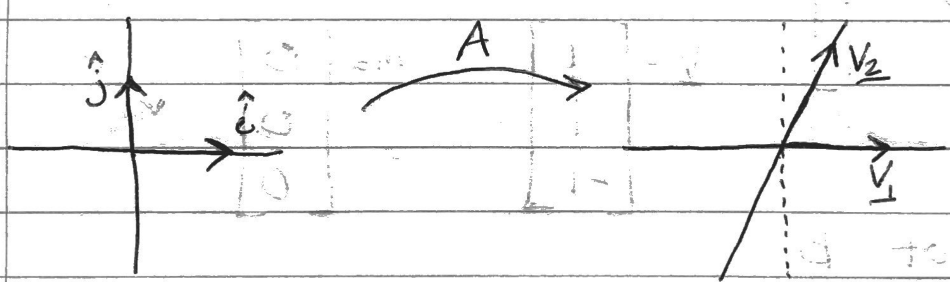
Typically, a matrix is viewed as a collection of values put into rows and columns.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

But matrices can also be viewed based on what they do.

Ex $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ $\hat{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\hat{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$A\hat{i} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \underline{v_1}$ $A\hat{j} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \underline{v_2}$



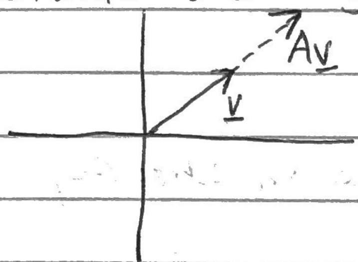
* Matrices act on vectors to produce different vectors.

A can also be called a "linear opperator" or a "linear transformation".

Eigenvectors: Is there a vector $\underline{v}$ that when acted on by a linear opperator produces a vector pointing in the same direction as $\underline{v}$?

$$A\underline{v} = \lambda \underline{v}$$

λ is a scalar value and is called the eigenvalue.
Eigenvalues and eigevectors exist as pairs.



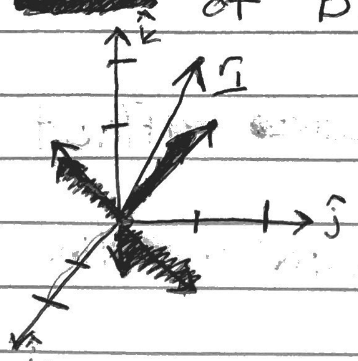
$$A\underline{v} = \lambda \underline{v}$$
$$A\underline{v} = \lambda I \underline{v} \quad (I \text{ is identity matrix})$$
$$A\underline{v} - \lambda I \underline{v} = \underline{0}$$
$$(A - \lambda I)\underline{v} = \underline{0}$$

* $\underline{v} = 0$ is a trivial solution. We want the non-trivial solutions. What must be true about $A - \lambda I$ for a non-trivial solution to exist?

Ex] B[redacted]= $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ $\quad B\underline{v} = \underline{m}$

$\underline{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad m = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ✓

* Plot [redacted] rows of B



[redacted] $\underline{r_1} + \underline{r_2} = \underline{r_3}$

* this means that B maps the 3-D space to the 2-D plane with normal vector $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

B has what we want: it maps a non-zero vector to the $\underline{0}$.

Properties of B: 1) The rows are scalar multiples of each other.

2) $\det(B) = 1\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} + 2\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$

$= 1(-1) - 1(1) + 2(1) \Rightarrow \det(B) = 0.$

So, need $\det(A - \lambda I) = 0$ for non-trivial $\underline{v}$

Ex $A = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$ $\qquad \det(A - \lambda I) = 0$

$\begin{vmatrix} 2-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} = 0$

$(2-\lambda)(1-\lambda) - (1) = 0 \Rightarrow 2 - 3\lambda + \lambda^2 - 1 = 0$

$\lambda^2 - 3\lambda + 1 = 0$

$\lambda = \dfrac{3 \pm \sqrt{9-4}}{2} \Rightarrow \boxed{\lambda = \frac{1}{2}(3 \pm \sqrt{5})}$

$\lambda = \frac{1}{2}(3 + \sqrt{5})$:

$\begin{bmatrix} 2 - \frac{3}{2} - \frac{\sqrt{5}}{2} & -1 \\ -1 & 1 - \frac{3}{2} - \frac{\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Rightarrow \begin{bmatrix} \frac{1}{2} - \frac{\sqrt{5}}{2} & -1 \\ -1 & -\frac{1}{2} - \frac{\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$\left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right)v_1 - v_2 = 0$ $\qquad -v_1 - \left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)v_2 = 0$

$v_1 = -\left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)$

$\boxed{\underline{v} = \begin{bmatrix} -\frac{1}{2}(1+\sqrt{5}) \\ 1 \end{bmatrix}}$ $\qquad \boxed{\lambda = \frac{1}{2}(3+\sqrt{5})}$

* Do the same thing for the other eigenvalue.

Eigen Decomposition: $A\underline{v} = \lambda \underline{v}$

Define: $V = [\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n]$ $\quad \Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & \\ 0 & & \ddots \lambda_n \end{bmatrix}$

$$AV = V\Lambda$$

Post Multiply both sides by V^{-1}

$$A(VV^{-1}) = V\Lambda V^{-1} \Rightarrow \boxed{A = V\Lambda V^{-1}}$$

This is very useful if you need to evaluate A^n

Ex $A^3 = (V\Lambda V^{-1})^3$

$$= (V\Lambda V^{-1})(V\Lambda V^{-1})(V\Lambda V^{-1})$$

$$= (V\Lambda V^{-1})(V\Lambda V^{-1} V\Lambda V^{-1})$$

$$= (V\Lambda V^{-1})(V\Lambda I\Lambda V^{-1})$$

$$= (V\Lambda V^{-1})(V\Lambda^2 V^{-1})$$

$$\boxed{A^3 \Rightarrow V\Lambda^3 V^{-1}}$$

$$\boxed{\Lambda^3 = \begin{bmatrix} \lambda_1^3 & 0 & \cdots & 0 \\ 0 & \lambda_2^3 & \cdots & 0 \\ 0 & \cdots & & \lambda_n^3 \end{bmatrix}}$$